Mechanics of Flexible Bodies in Local Orthonormal Frames

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Configurations of a flexible body In this work, we treat flexible bodies as Riemannian manifolds, namely smooth manifolds equipped with an inner product embodied by the metric tensor. To describe the geometry of a flexible body, material coordinates $\bar{\alpha} = (\alpha^1,\ \alpha^2,\ \alpha^3)$ are introduced on a reference configuration of the body and a continuous parameter t is used to refer to successive current configurations of the body over time.

Coordinate expression of vector fields and of their derivatives The equations of motion of a body involves the derivatives of vector fields, from which most of the computational complexity stems. A vector field $\vec{w}(\bar{\alpha}, t)$ of the current configuration can be resolved the local base vectors of the current configuration and differentiated with respect to $\star$ (including time and space coordinates) as

$$\vec{w}(\bar{\alpha}, t) = [{}^X\underline{w}(\bar{\alpha}, t)]^I\ \vec{g\mathcal{x}}_I(\bar{\alpha}, t) \quad \rightarrow \quad d_\star\vec{w}(\bar{\alpha}, t) = [d_\star\, {}^X\underline{w}]^I\ \vec{g\mathcal{x}}_I + [{}^X\underline{w}]^I\ d_\star\vec{g\mathcal{x}}_I \tag{1}$$

where ${}^X\underline{w}(\bar{\alpha}, t)$ are the coordinates of $\vec{w}(\bar{\alpha}, t)$ in the (covariant) base vectors $\vec{g\mathcal{R}}_I(\bar{\alpha})$ of the current configuration. Because the base vectors vary in space and in time, their derivatives contribute to the expression. Following standard derivations of Riemannian geometry, the derivatives of the base vectors are denoted $d_J\vec{g\mathcal{x}}_I = \Gamma^K_{IJ}\ \vec{g\mathcal{x}}_K$ where $\Gamma^K_{IJ}(\bar{\alpha}, t)$ are the so-called Christoffel symbols of the second kind. Eventually, the derivative of a vector field becomes

$$d_\star\vec{w}(\bar{\alpha}, t) = [D_\star\, {}^X\underline{w}]^I\ \vec{g\mathcal{x}}_I \tag{2}$$

where $D_\star\, {}^X\underline{w}(\bar{\alpha}, t)$ are the coordinates of the *covariant derivative* of $\vec{w}$ with respect to $\star$.

Local orthonormal frames Because the space in which a body exists is locally Euclidean, an alternative strategy to resolve vector fields in coordinates is to introduce an arbitrary field of local orthonormal bases $B(\bar{\alpha}, t)$ made of the three basis vectors

$$\vec{B}_i(\bar{\alpha}, t) \quad \text{with} \quad \vec{B}_i(\bar{\alpha}, t)\cdot\vec{B}_j(\bar{\alpha}, t) = \delta_{ij} \tag{3}$$

Orthonormal bases are elements of $SO(3)$, the Special Orthogonal group. The pairs (material point, local orthonormal basis) are referred to as *local orthonormal frames*. A vector field $\vec{w}(\bar{\alpha}, t)$ of the current configuration can be resolved the local orthonormal bases and differentiated with respect to $\star$ (including time and space coordinates) as

$$\vec{w}(\bar{\alpha}, t) = [{}^B\underline{w}(\bar{\alpha}, t)]_i\ \vec{B}_i(\bar{\alpha}, t) \quad \rightarrow \quad d_\star\vec{w}(\bar{\alpha}, t) = [d_\star\, {}^B\underline{w}]_i\ \vec{B}_i + [{}^B\underline{w}]_i\ d_\star\vec{B}_i \tag{4}$$

where ${}^B\underline{w}(\bar{\alpha}, t)$ are the coordinates of $\vec{w}(\bar{\alpha}, t)$ in the local orthonormal bases $B(\alpha, t)$. Because the local orthonormal basis vectors vary in space and in time, their derivatives contribute to the expression. Due to the orthonormality relationship, differentiation with respect to $\star$ yields $d_\star\vec{B}_i(\bar{\alpha}, t) = [{}^B\underline{\underline{\tilde{\star}}}_B]_{ij}\ \vec{B}_j$ with $[{}^B\underline{\underline{\tilde{\star}}}_B]_{ij} = -[{}^B\underline{\underline{\tilde{\star}}}_B]_{ji}$ where ${}^B\underline{\underline{\tilde{\star}}}_B(\bar{\alpha}, t)$ are skew-symmetric matrices which can be seen as the local components of skew-symmetric tensor fields $\tilde{\star}(\bar{\alpha}, t) = [{}^B\underline{\underline{\tilde{\star}}}_B]_{ij}\vec{B}_i \otimes \vec{B}_j$. These skew-symmetric tensor fields are elements of $\mathfrak{so}(3)$, the Lie algebra of $SO(3)$. The skew-symmetric tensor fields associated with the differentiation with respect to space and time are denoted ${}^B\tilde{\kappa}_{I,B}$ and ${}^B\tilde{\omega}_B$, respectively. Eventually, the derivative of a vector field becomes

$$d_\star\vec{w}(\bar{\alpha}, t) = \left[D_\star\, {}^B\underline{w}\right]_i\ \vec{B}_i \tag{5}$$

where $D_\star\, {}^B\underline{w} = d_\star\, {}^B\underline{w} + {}^B\underline{\underline{\tilde{\star}}}_B\, {}^B\underline{w}$ are the coordinates of the *gauge-covariant derivative* of $\vec{w}$ with respect to $\star$.

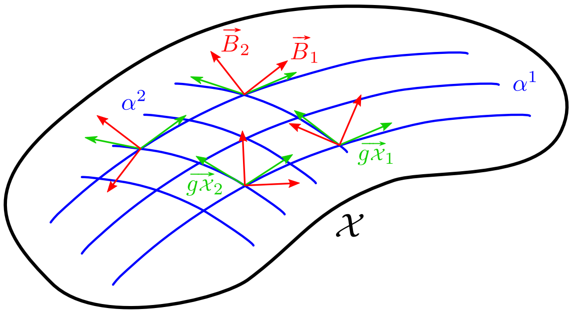

Figure 1: Local orthonormal bases $B(\bar{\alpha}, t)$ made of basis vectors $\vec{B}_i(\bar{\alpha}, t)$ introduced on the current configuration $\mathcal{X}$ of a flexible body. The convected coordinate lines are denoted by α and the base vectors tangent to these coordinate lines are denoted $\overrightarrow{g\mathcal{X}}_I(\bar{\alpha})$.

Mechanics of a flexible body The vector form of the equations of motion of a flexible body can be derived from variational principles. The three equilibrium equations in the volume of a body are $d_I \vec{P}^I - \rho\, d_t \vec{v} + \vec{f} = \vec{0}$ where $\vec{P}^I(\bar{\alpha}, t)$ are the (contra-variant) traction vectors associated with the strain energy of the body, the second term are the inertia forces derived from the kinetic energy of the body, with $\rho(\bar{\alpha})$ the mass density, and $\vec{f}(\bar{\alpha}, t)$ are the externally applied body forces such as gravity. The coordinate expression of these equilibrium equations resolved in the local orthonormal bases is given by

$$D_I\, {}^B\underline{P}^I - \rho\, D_t\, {}^B\underline{v} + {}^B\underline{f} = \underline{0} \Leftrightarrow \left(d_I\, {}^B\underline{P}^I + {}^B\tilde{\kappa}_{I,B}\, {}^B\underline{P}^I\right) - \rho\left(d_t\, {}^B\underline{v} + {}^B\tilde{\omega}_B\, {}^B\underline{v}\right) + {}^B\underline{f} = \underline{0} \qquad (6)$$

Clearly, the derivatives of the local orthonormal bases interact with the mechanical quantities, namely the velocity and the internal stresses. Nevertheless, no governing equations for them are obtained from the fundamental principles of mechanics. That is consistent with the arbitrary nature of the local orthonormal bases - if governing equations were obtained, the bases would in fact not be arbitrary.

Field of local orthonormal bases as a gauge The local orthonormal bases and the related transformation between them form a *gauge group* and their derivatives are so-called *gauge fields*. This framework arises from the invariance, or symmetry, of the Lagrangian of the body under a *local* change of coordinates.
A gauge introduces additional, redundant degrees of freedom in the description of a flexible body. The mathematical procedure that deals with the elimination of the redundancy is referred to as *gauge fixing*. This is done typically by providing *a priori* governing equations for the redundant degrees of freedom or by adding gauge breaking terms in the form of energy-like contributions associated with the gauge fields. Procedures akin to gauge fixing are encountered for instance in the floating frame of reference [1], where a time dependent orthonormal base is used for an entire body and the inertia of the undeformed body is associated to the rotational velocity of that base. Another example is the use of regularization terms to enforce the polar decomposition of the deformation gradient [2], namely that the local bases are such that the local coordinates of the deformation gradient are symmetric.

Contributions The framework that we present hopefully turns out to be well suited for the swift introduction of common kinematic approximations such as rigid-body, beam, and plate/shell, and for the design of efficient numerical methods such as geometric time integration and finite element discretization. Gauge fixing procedures open the door to the simplification of solution processes and the enrichment of approximation methods.

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References
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