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Dynamic Analysis of Naturally Curved Rods using Cosserat Rod Theory

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Abstract

This paper is aimed to develop and solve the continuous Cosserat dynamic equations of motion for rods with uniform intrinsic curvatures in the Kirchhoff's framework of unshearability and inextensibility. The initial-boundary value problem posed by the system of dynamic equations of the rod are solved using a stable numerical method. A case study of a helical spring will be presented, where its dynamic behavior using the numerical solution will be compared with the solution from a commercial finite element software.

A Cosserat rod which is thin in two dimensions can be considered as a one-dimensional material curve, where every material point is attached with a local coordinate system composed of vectors known as directors. A comprehensive description of the Cosserat rod theory can be found in [1], which can incorporate bending, torsion, extension are shear deformations. The centerline position and the directors denote the evolution of the cross-section of the rod in space, as depicted in Figure 1.

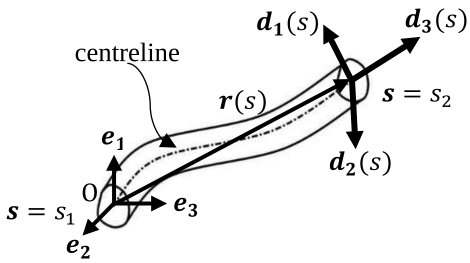

Figure 1: A Cosserat rod showing centerline and directors

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ represents the global frame. The centerline is parameterized by the arc-length parameter '**s**'. $\boldsymbol{r}(s)$ denotes the position of the centerline and the orthonormal triad $\{\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s)\}$ represents the directors at any point along the centerline. $\boldsymbol{R}(s) \in SO(3)$ represents the cross-section orientation. Thus, a Cosserat rod is characterized by the centerline position and the cross-section orientation. The unstrechability can be ensured by imposing the condition that the director $\boldsymbol{d}_3$ is along the tangent to the centerline. The condition of unstretchability is given in Equation (1). Let $(\,)'$ represents the derivative with respect to 's' and $(\dot{\,})$ or $\partial_t(\,)$ represents the derivative with respect to time. The operator $\widehat{(\,)}$ denotes the cross-product matrix.

$$\frac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|} - \boldsymbol{d}_3 = 0 \tag{1}$$

If the rod is assumed to be unshearable, the cross-section normal must lie along the centerline tangent. The unshearability conditions are given in Equation (2) and Equation (3).

$$\boldsymbol{d}_1 \cdot \boldsymbol{r}' = 0 \tag{2}$$

$$\boldsymbol{d}_2 \cdot \boldsymbol{r}' = 0 \tag{3}$$

With the constraints given in Equations (1), (2) and (3), the configuration of any cross-section on the rod can be defined using three parameters. The spatial evolution of the centerline and the directors are represented by the following relations in Equation (4) and Equation (5).

$$\boldsymbol{r}' = \boldsymbol{R}\boldsymbol{v} \tag{4}$$

$$\boldsymbol{R}' = \boldsymbol{R}\widehat{\boldsymbol{k}} \tag{5}$$

In the above equations, vector $\boldsymbol{v}$ represents shear strains and stretch which accounts for the rate of change of the centerline expressed in the director frame. The rate of change of the cross-section orientation expressed in the director frame is defined using the vector $\boldsymbol{k}$ which represents bending curvatures and the twist. The dynamic equations of motion of the rod obtained by the linear and angular momentum balance on an arbitrary section of the rod are a set of hyperbolic partial differential equations (PDEs) and are given in Equation (6) and Equation (7).

$$\boldsymbol{n}' + \widehat{\boldsymbol{n}} = \boldsymbol{\rho A}\ddot{\boldsymbol{r}} \tag{6}$$

$$\boldsymbol{m}' + \boldsymbol{r}' \times \boldsymbol{n} + \widehat{\boldsymbol{m}} = \partial_t(\boldsymbol{R\rho J\omega}) \tag{7}$$

In Equation (4) and Equation (5), $\boldsymbol{n}$ and $\boldsymbol{m}$ are respectively the point forces and moments acting on the rod, and $\widehat{\boldsymbol{n}}$ and $\widehat{\boldsymbol{m}}$ are respectively the distributed force and moment per unit length acting on the rod. $\boldsymbol{\rho}$ is the mass density of the rod and $\boldsymbol{A}$ is the area of cross-section of the rod. $\boldsymbol{J}$ is the matrix containing the second moments of area of the cross-section. $\boldsymbol{\omega}$ is the angular velocity of the director frame. The strains and curvatures at the deformed configuration are related to the applied forces and moments by the linear constitutive laws given in Equation (8) and Equation (9).

$$\boldsymbol{n} = \boldsymbol{R}\boldsymbol{C}_{\boldsymbol{se}}(\boldsymbol{v} - \boldsymbol{v}^*) \tag{8}$$

$$\boldsymbol{m} = \boldsymbol{R}\boldsymbol{C}_{\boldsymbol{bt}}(\boldsymbol{k} - \boldsymbol{k}^*) \tag{9}$$

$\boldsymbol{C}_{\boldsymbol{se}}$ is a matrix containing shear and stretching stiffnesses and $\boldsymbol{C}_{\boldsymbol{bt}}$ is a matrix of bending and torsional stiffnesses. $\boldsymbol{v}^*$ and $\boldsymbol{k}^*$ are the reference values of strains and curvature based on the initial shape of the rod.

An explicit time marching scheme is used as used [2], which requires small time-step to ensure the stability of the algorithm. An implicit time differentiation scheme is followed [3], resulting in a system of ordinary differential equations (ODE's) which can be solved as boundary value problem (BVP).

References

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